International Journal of Theoretical Physics, Vol. 15. No. 11 (1976), pp. 815–823

Axially Symmetric Electrovac Solutions

S. BANERJI

Department of Physics, The University of Burdwan, Burdwan, India

and

K. C. DAS

Department of Physics, Katwa College, Katwa, Burdwan, India

Received: 1 *July* 1975

Abstract

Two classes of electrovac solutions are obtained in oblate spheroidal coordinates, which are the electromagnetic analogs of Zipoy's monopole and dipole solutions. The asymptotic behavior of the solutions is studied to gain some insight into the nature of the source of the gravitational and electromagnetic fields. A similar stationary solution of the pure gravitational field is found to belong to Papapetrou's class.

1. *Introduction*

The equilibrium shape of a rotating star is an oblate spheroid. Hence axially symmetric solutions of Einstein's field equations are of interest in astrophysics. Misra (1960) used oblate spheroidal coordinates to obtain static solutions of sourceless Einstein's equations for the gravitational field. The starting point of his investigation was the familiar axially symmetric static line element in Weyl canonical form:

$$ds^2 = -e^{2(\lambda-\sigma)}(d\rho^2 + dz^2) - \rho^2 e^{-2\sigma} d\phi^2 + e^{2\sigma} dt^2 \tag{1.1}$$

where

$$\lambda = \lambda(\rho, z), \qquad \sigma = \sigma(\rho, z)$$

Oblate spheroidal coordinates (u, θ) are then introduced with the help of the transformation

$$\rho = a \cosh u \cos\theta, \qquad z = a \sinh u \sin\theta \tag{1.2}$$

The metric (1.1) then reduces to the form

$$ds^2 = -a^2 e^{2(\lambda-\sigma)}(\sinh^2 u + \sin^2\theta)(du^2 + d\theta^2) - a^2 e^{-2\sigma} \cosh^2 u \cos^2\theta\, d\phi^2 \tag{1.3}$$

where

$$0 \leqslant u < \infty, \qquad -\pi/2 \leqslant \theta \leqslant \pi/2, \qquad 0 \leqslant \phi \leqslant 2\pi, \quad -\infty < z < \infty$$

Zipoy (1966) showed that solutions for "Newtonian potential" σ can be written as a linear combination of Legendre poynomials of integral order l. He discussed solutions corresponding to $l = 0$, $l = 1$, and a combination $l = 0$, 1 and endowed them with complicated topological properties. Bonnor and Sackfield (1968) showed that the solutions for $l = 0$, $l = 1$ can be interpreted as due to mass monopole and dupole, respectively, with Euclidean topology.

The only astronomical objects discovered so far where general relativistic effects are not negligible are pulsars. They are rotating stars with a large magnetic field. The angular velocities are small in relativistic units but the magnetic field is high. Hence axially symmetric solutions of Einstein-Maxwell equations have attracted a lot of attention. Misra (1962) found some electrovac solutions in oblate spheroidal coordinates. Bonnor (1961) gave methods for generating axially symmetric electrovac solutions from the corresponding solutions of $R_{\alpha\beta} = 0$.

In sections 2 and 3 we discuss some static electrovac solutions in oblate spheroidal coordinates, which can be regarded as electromagnetic analogs of Zipoy's solutions (1966). In section 4 we find a similar empty space stationary solution of the gravitational field.

2. *A Class of Conformastat Solutions*

Let us take a special case of the metric (1.1) in the conformastat form:

$$ds^2 = -a^2 e^{-2\sigma}[(\sinh^2 u + \sin^2 \theta)(du^2 + d\theta^2) + \cosh^2 u \cos^2 \theta \, d\phi^2] + e^{2\sigma} \, dt2 \tag{2.1}$$

We can now write the Einstein-Maxwell equations for empty space in the form

$$R_{\alpha\beta} = -8\pi E_{\alpha\beta} \tag{2.2}$$

$$E_\beta{}^\alpha = \tfrac{1}{4}\delta_\beta{}^\alpha F^{\mu\nu}F_{\mu\nu} - F^{\alpha\mu}F_{\beta\mu} \tag{2.3}$$

$$F_{\alpha\beta;\nu} + F_{\beta\nu;\alpha} + F_{\nu\alpha;\beta} = 0 \tag{2.4}$$

$$F^{\alpha\beta}_{;\beta} = 0 \tag{2.5}$$

We can find the following solutions analogous to those for pure gravitation field found by Zipoy (1966).

Case (a) $l = 0$ – Monopole Solution

$$e^{2\sigma} = [1 + 8\pi(A^2 + B^2)^{1/2}\beta \tan^{-1}(\operatorname{cosech} u)]^{-2}, \quad 0 \leqslant \tan^{-1}(\operatorname{cosech} u) \leqslant \pi/2 \tag{2.6}$$

$$F_{41} = 2B\beta \operatorname{sech} u [1 + 8\pi(A^2 + B^2)^{1/2}\beta \tan^{-1}(\operatorname{cosech} u)]^{-2} \tag{2.7}$$

$$F_{23} = -2Aa\beta \cos\theta \tag{2.8}$$

where A, B, and β are constants. If we put $A = 0$ this reduces to a pure electric field while if $B = 0$ it is a pure magnetic field. The constant of integration in (2.6) is chosen in such a way that the metric reduces to the Euclidean form at spatial infinity. The above metric can also be deduced from Zipoy's solution (1966) corresponding to $l = 0$ by using Bonnor's theorem (1961).

Spherical polar coordinates $(r, \bar{\theta})$ are connected with the cylindrical coordinates (ρ, z) by means of the relations:

$$r = (\rho^2 + z^2)^{1/2} \qquad \text{and} \qquad \bar{\theta} = \sin^{-1} z/r \tag{2.9}$$

Contrary to the usual convention $\bar{\theta}$ is here measured from the equator rather than the pole for comparison with oblate spheroidal coordinates. Using equation (1.2) we obtain

$$r = a(\sinh^2 u + \cos^2 \theta)^{1/2} \xrightarrow[u\to\infty]{} a \sinh u \tag{2.10a}$$

$$\bar{\theta} = \sin^{-1} \frac{a \sinh u \sin \theta}{r} \xrightarrow[u\to\infty]{} \theta \tag{2.10b}$$

Hence at large distances we can write

$$e^{2\sigma} = [1 + 8\pi(A^2 + B^2)^{1/2}\beta \tan^{-1}(a/r)]^{-2} \tag{2.11}$$

The metric (2.1) reduces at large distances to the isotropic form of the Schwarzschild solution in spherical polar coordinates with

$$m = 8\pi(A^2 + B^2)^{1/2}\beta a \tag{2.12}$$

If we put $A = 0, B = 0$, then $m = 0$. This may mean either of two things: (i) The mass of the source is related in such a way to the source of electric and magnetic fields that the former vanishes with the vanishing of the latter, or (ii) the source does not have a mass of its own but only "electromagnetic mass," i.e., mass derived from the energy of the electromagnetic field. We further notice that the space-time becomes flat with the vanishing of the electromagnetic field ($A = B = 0$).

We know that u = const. surfaces are oblate spheroids and the above solution depends on u alone. Hence this has spheroidal symmetry. It should be noted that θ is discontinuous as θ = const. lines cross $u = 0$. Therefore $\partial g_{\mu\nu}/\partial z$ is discontinuous over the disk $z = 0, \rho \leqslant a$.

Bonnor and Sackfield (1968) discussed Zipoy's solutions (1966) for the pure gravitational field. They interpreted the discontinuity of $(\partial\sigma/\partial z)$ and σ across the disk as the presence of a monopole and a dipole layer, respectively. Using Newtonian potential theory they calculated the strength of these monopole and dipole layers from the magnitudes of the discontinuities. All this was possible because σ satisfies Laplace's equation for the pure gravitational field. But if we use the same kind of interpretation in our electrovac solutions then we shall be led to contradictions because here σ does not satisfy Laplace's equation. However, for the conformastat metric $e^{-\sigma}$ always satisfies Laplace's equation in the electrovac case (Synge, 1960).

We now introduce electric and magnetic potentials χ and ψ respectively according to the equations

$$F_{4a} = \chi_{;a}, \qquad F_{ab} = (-g)^{1/2}\epsilon_{abc}\psi^{;c} \tag{2.13}$$

where ϵ_{abc} is the Levi-Civita permutation symbol.

From equations (2.7) and (2.8) we obtain

$$\chi = (4\pi)^{-1}(A^2 + B^2)^{-1/2}B\{[1 + 8\pi(A^2 + B^2)^{1/2}\beta \tan^{-1}(\operatorname{cosech} u)]^{-1} - 1\} \tag{2.14}$$

$$\psi = -2A\beta \tan^{-1}(\operatorname{cosech} u) \tag{2.15}$$

It should be noted that A is present in the expression for χ, i.e., the presence of the magnetic field affects the electric potential but not vice verse. The constants of integration have been chosen such that at $r \to \infty$, $\chi \to 0$, and $\psi \to 0$. $\partial\chi/\partial z$ and $\partial\psi/\partial z$ are discontinuous over the disc $z = 0$, $\rho \leqslant a$. Taking the asymptotic expansion of χ for large values of r we have

$$\chi = -2B\beta a/r + O(r^{-2}) \tag{2.16}$$

We know that inside matter Maxwell's equation is

$$F^{\mu\nu}_{;\nu} = j^{\mu} \tag{2.17}$$

For

$$\mu = 4, \qquad (F^{4\nu}\sqrt{-g})_{,\nu} = j^4\sqrt{-g} \tag{2.17a}$$

Using Gauss's theorem and integrating both sides over the volume of an infinite sphere we obtain the total charge of the source:

$$e = -8\pi B\beta a \tag{2.18}$$

In the absence of the magnetic field, $A = 0$, (2.12) reduces to

$$m = \pm 8\pi B\beta a \tag{2.12a}$$

Hence in such a case

$$e = \pm m \tag{2.19}$$

Bonnor and Wickramasuriya (1972) used this solution and matched it with the solution inside an oblate spheroid containing charged dust with charge density equal to the matter density.

Taking the asymptotic expansion of the magnetic potential at $r \to \infty$ we find that the monopole term $O(r^{-1})$ is present.

Case (b) l = 1 – Dipole Solution

$$e^{2\sigma} = \{1 - 8\pi(A^2 + B^2)^{1/2}\gamma[1 - \sinh u \tan^{-1}(\operatorname{cosech} u)] \sin\theta\}^{-2} \tag{2.20}$$

$$F_{41} = 2Be^{2\sigma}\gamma \sin\theta\,[\tanh u - \cosh u \tan^{-1}(\operatorname{cosech} u)] \tag{2.21}$$

$$F_{42} = 2Be^{2\sigma}\gamma \cos\theta\,[1 - \sinh u \tan^{-1}(\operatorname{cosech} u)] \tag{2.22}$$

$$F_{23} = -Aa\gamma \sin 2\theta\,[\sinh u - \cosh^2 u \tan^{-1}(\operatorname{cosech} u)] \tag{2.23}$$

$$F_{31} = -2Aa\gamma \cosh u \cos^2\theta\,[1 - \sinh u \tan^{-1}(\operatorname{cosech} u)] \tag{2.24}$$

$$\chi = -(4\pi)^{-1}(A^2 + B^2)^{-1/2}B \times (\{1 - 8\pi(A^2 + B^2)^{1/2}\gamma[1 - \sinh u \tan^{-1}(\operatorname{cosech} u)]\sin\theta\}^{-1} - 1) \tag{2.25}$$

$$\psi = 2A\gamma \sin\theta\,[1 - \sinh u \tan^{-1}(\operatorname{cosech} u)] \tag{2.26}$$

At large distances $e^{2\sigma}$ reduces to the form

$$e^{2\sigma} = \{1 - 8\pi(A^2 + B^2)^{1/2}\gamma\left[1 - \frac{r}{a}\tan^{-1}\left(\frac{a}{r}\right)\right]\sin\bar{\theta}\}^{-2} \tag{2.27}$$

Expanding this expression in powers of $1/r$ we find that the term in $1/r$ is abser while the coefficient of the term in $1/r^2$ is $(8\pi a^2/3)(A^2 + B^2)^{1/2}\gamma \sin\bar{\theta}$. This shows that the mass of the source is zero while it has dipole moment. Since a dipole contains equal quantities of positive and negative mass, the total mass vanishes. Here $g_{\mu\nu}$ is discontinous across the disk $z = 0$, $\rho \leqslant q$ and so also is $\partial g_{\mu\nu}/\partial z$. As in the case of total mass in case (a), the dipole moment in this case vanishes when $A = B = 0$ and the line element assumes the Minkowskian form.

Similarly the asymptotic expansion of χ and ψ reveal the existence of electric and magnetic dipole moments of the source but no monopoles. Not only are χ and ψ themselves discontinuous across the disk but so also is $\partial\chi/\partial z$.

Case (c) $l = 0$ and 1 Combined

$$e^{2\sigma} = (1 + 8\pi(A^2 + B^2)^{1/2}\{\beta \tan^{-1}(\operatorname{cosech} u) - \gamma[1 - \sinh u \tan^{-1}(\operatorname{cosech} u)]\sin\theta\})^{-2} \tag{2.28}$$

$$F_{41} = 2Be^{2\sigma}\{\beta \operatorname{sech} u + \gamma \sin\theta\,[\tanh u - \cosh u \tan^{-1}(\operatorname{cosech} u)]\} \tag{2.29}$$

Similarly the other components of $F_{\mu\nu}$ can be written. The electric and magnetic potentials will evidently be suitable combinations of the values in cases (a) and (b). Hence the monopole and dipole moments will also be combinations of these quantities in the two cases.

3. *Another Class of Solutions*

We took above only the special case of the metric (1.3) when $\lambda = 0$. Bonnor (1961) gave us a theorem for generating electrovac solutions from empty space solutions of the pure gravitational field even when $\lambda \neq 0$. We can therefore generate a more general class of solutions from those of Zipoy (1966)

Case (a) l = 0 – Monopole Solution

$$e^{2\sigma} = [16\pi(A^2 + B^2)]^{-1} \operatorname{cosech}^2 [-\beta \tan^{-1}(\operatorname{cosech} u) + C] \tag{3.1}$$

where

$$C = \sinh^{-1}[16\pi(A^2 + B^2)]^{-1/2} \tag{3.2}$$

$$e^{2\lambda} = \left(\frac{\sinh^2 u + \sin^2 \theta}{\cosh^2 u}\right)^{\beta^2} \tag{3.3}$$

$$F_{41} = [8\pi(A^2 + B^2)]^{-1} B\beta \operatorname{sech} u \operatorname{cosech}^2 [-\beta \tan^{-1}(\operatorname{cosech} u) + C] \tag{3.4}$$

$$F_{23} = -2Aa\beta \cos\theta \tag{3.5}$$

$$\chi = [8\pi(A^2 + B^2)]^{-1} \{\coth C - \coth[-\beta \tan^{-1}(\operatorname{cosech} u) + C]\} \tag{3.6}$$

$$\psi = -2A\beta \tan^{-1}(\operatorname{cosech} u) \tag{3.7}$$

It can be shown, as in the corresponding case in the previous section, that the metric reduces at large distances to that of Schwarzschild with

$$m = -a\beta \coth C \tag{3.8}$$

$\coth C$ can be evaluated from equation (3.2). β and C must be given opposite signs to make m positive.

The above metric has spheroidal symmetry similar to that given by equation (2.6). Further, $\partial g_{\mu\nu}/\partial z$, $\partial\chi/\partial z$, and $\partial\psi/\partial z$ are all discontinuous across the disk. The asymptotic expansions of both χ and ψ show the existence of monopoles. As in case (a) of the last section the total electric charge of the source is given by

$$e = -[2(A^2 + B^2)]^{-1} a\beta \operatorname{cosech}^2 C = -8\pi a\beta \tag{3.9}$$

i.e.,

$$e/m = 8\pi \tanh C \tag{3.10}$$

If we put $A = 0, B = 0$, then the solution does not exist, unlike in case (a) of the last section. At $u = 0$, $\theta = 0$, i.e., $\rho = a$, $z = 0$ (the rim of the disk referred to above), $e^{2\lambda} = 0$. This is an essential singularity of the metric. The nature of the singularity has been discussed in detail by Zipoy (1966).

Case (b) l = 1 – Dipole Solution

$$e^{2\sigma} = [16\pi(A^2 + B^2)]^{-1} \operatorname{cosech}^2 \{\gamma[1 - \sinh u \tan^{-1}(\operatorname{cosech} u)] \sin\theta + C\} \tag{3.11}$$

where C is given by equation (3.2).

$$e^{2\lambda} = \left(\frac{\sinh^2 u + \sin^2 \theta}{\cosh^2 u}\right)^{-\gamma 2} \exp(-\gamma^2 \cos^2 \theta \{[\tan^{-1}(\operatorname{cosech} u)]^2 + [1 - \sinh u \tan^{-1}(\operatorname{cosech} u)]^2\}) \tag{3.12}$$

F_{41}, F_{42}, F_{23}, and F_{31} are given by equations (2.21)-(2.24) with $e^{2\sigma}$ given by (3.11). Further

$$\chi = [8\pi(A^2 + B^2)]^{-1} B(\coth C - \coth\{\gamma[1 - \sinh u \tan^{-1}(\operatorname{cosech} u)] \sin\theta + C\}) \tag{3.13}$$

ψ is given by equation (2.26).

Taking the asymptotic expansion of (3.11) we find that the total mass of the source is zero and the coefficient of $1/r^2$ is $-\frac{2}{3}\gamma a^2 \coth C \sin\bar{\theta}$, indicating the presence of dipole moment. The asymptotic expansions of χ and ψ show the existence of the electric and magnetic dipole moments, but there are no monopoles. χ, $\partial\chi/\partial z$, and ψ are discontinuous across the disk.

Case (c) l = 0 and 1 (Combined)

$$e^{2\sigma} = [16\pi(A^2 + B^2)]^{-1} \operatorname{cosech}^2\{-\beta \tan^{-1}(\operatorname{cosech} u) + \gamma[1 - \sinh u \tan^{-1}(\operatorname{cosech} u)] \sin\theta + C\} \tag{3.14}$$

$$e^{2\lambda} = \left(\frac{\sinh^2 u + \sin^2 \theta}{\cosh^2 u}\right)^{\beta^2 - \gamma^2} \exp(-4\beta\gamma[\sin\theta \tan^{-1}(\operatorname{cosech} u) - \tan^{-1}(\operatorname{cosech} u \sin\theta)] - \gamma^2 \cos^2\theta \{[\tan^{-1}(\operatorname{cosech} u)]^2 + [1 - \sinh u \tan^{-1}(\operatorname{cosech} u)]^2\}) \tag{3.15}$$

F_{41} is given by equation (2.29) with $e^{2\sigma}$ substituted from (3.14). The other components of $F_{\mu\nu}$, χ, and ψ can be similarly written. On account of the nonlinearity of the field equations the expression for $e^{2\lambda}$ in (3.15) is not just the sum of (3.3) and (3.12) but an additional "interference term" occurs. This gives rise to additional singularities discussed in detail by Zipoy (1966).

4. *Stationary Solutions of Gravitational Field*

Ehlers (1959, 1962) and later Misra and Pandey (1971) gave methods for generating a stationary solution of a source-free gravitational field from a static solution. Bonnor (1961) pointed out the formal similarity between these methods and the purely magnetic field solutions found in cases (a) and (b) of the last section with $B = 0$. By these methods we obtain a metric of the form

$$ds^2 = -a^2 e^{2(\lambda - \sigma)}(\sinh^2 u + \sin^2\theta)(du^2 + d\theta^2) - a^2 e^{-2\sigma} \cosh^2 u \cos^2\theta \, d\phi^2 + e^{2\sigma}(dt - H\, d\phi)^2 \tag{4.1}$$

We generate from Zipoy's solutions the following:

Case (a) l = 0

$$e^{2\sigma} = A^{-1} \operatorname{sech}[-2\beta \tan^{-1}(\operatorname{cosech} u) + k_1] \tag{4.2}$$

$$e^{2\lambda} = \left(\frac{\sinh^2 u + \sin^2 \theta}{\cosh^2 u}\right)^{\beta^2} \tag{4.3}$$

$$H = 2Aa\beta \sin\theta + C_1 \tag{4.4}$$

Case (b) l = 1

$$e^{2\sigma} = A^{-1} \operatorname{sech}\{2\gamma[1 - \sinh u \tan^{-1}(\operatorname{cosech} u)] \sin\theta + k_2\} \tag{4.5}$$

$$e^{2\lambda} = \left(\frac{\sinh^2 u + \sin^2 \theta}{\cosh^2 u}\right)^{-\gamma^2} \exp(-\gamma^2 \cos^2\theta \{[\tan^{-1}(\operatorname{cosech} u)]^2 + [1 - \sinh u \tan^{-1}(\operatorname{cosech} u)]^2\}) \tag{4.6}$$

$$H = Aa\gamma \cos^2\theta\, [\cosh^2 u \tan^{-1}(\operatorname{cosech} u) - \sinh u] + C_2 \tag{4.7}$$

K_1, K_2, C_1, and C_2 are constants of integration. Similarly the metric for the combined case $l = 0, 1$ can be written. All of them have a singularity at $u = 0$, $\theta = 0$. None of them go over to the Euclidean form at spatial infinity.

It is relevant to mention here that all the stationary solutions generated by the method of Misra and Pandey (1971) are members of the class first discovered by Papapetrou (1953) [Levy (1972)]. Papapetrou (1953) has shown that for this class the term of order $(1/r)$ in the asymptotic expansion of g_{44} is absent if we impose the condition of asymptotic flatness. This means that the source has angular momentum but no mass. If, however, the condition of asymptotic flatness is relaxed we can retain the term of order $(1/r)$ as in the above cases.

Acknowledgment

Our thanks are due to Professor A. K. Raychaudhuri for suggesting improvements in the paper.

References

Bonnor, W. B. (1961). *Zeitschrift für Physik*, **161**, 439.

Bonnor, W. B., and Sackfield, A. (1968). *Communications in Mathematical Physics*, 8, 338.

Bonnor, W. B., and Wickramasuriya, S. B. P. (1972). *International Journal of Theoretical Physics*, 5, 371.

Ehlers, J. (1962). *Bulletin of the American Physical Society*, 9, 425.

Ehlers, J. (1959). Proceedings of the Royaumount Conference, pp. 275–84 (CNRS, Paris).

Levy, H. (1972). *Physics Letters*, **42A**, 176.
Misra, M. (1960). *Proceedings of the National Institute of Science* (India), A26, 673.
Misra, M. (1962). *Proceedings of the National Institute of Science* (India), A28, 105.
Misra, R. M., and Pandey, D. B. (1971). *Physics Letters*, **34A**, 245.
Papapetrou, A. (1953). *Annalen der Physik*, **12**, 309.
Synge, J. L. (1960). *Relativity: The General Theory* (North-Holland), p. 369.
Zipoy, D. M. (1966). *Journal of Mathematical Physics*, **7**, 1137.